The fifteenth-seventeenth century transformation of abbacus algebra

Perhaps – though not thought of by Edgar Zilsel and Joseph Needham – the best illustration of the 'Zilsel-Needham thesis'

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Jens Høyrup
Roskilde University
Section for Philosophy and Science Studies
http://www.akira.ruc.dk/~jensh

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Erik Stinus *in memoriam*

Abstract

In 1942, Edgar Zilsel proposed that the sixteenth–seventeenth-century emergence of Modern science was produced neither by the university tradition, nor by the Humanist current of Renaissance culture, nor by craftsmen or other practitioners, but through an interaction between all three groups in which all were indispensable for the outcome. He only included mathematics via its relation to the "quantitative spirit". The present study tries to apply Zilsel's perspective to the emergence of the Modern algebra of Viète and Descartes (etc.), by tracing the reception of algebra within the Latin-Universitarian tradition, the Italian abbacus tradition, and Humanism, and the exchanges between them, from the twelfth through the late sixteenth and early seventeenth century.

Edgar Zilsel and the Zilsel-Thesis

Edgar Zilsel was an Austrian sociologist belonging to the circle of logical empiricists – together with Otto Neurath and Jørgen Jørgensen one of those who believed in the possibility of achieving reliable knowledge about the external world (and together with Neurath and Jørgensen closer to Marxism than most members of the movement).[1] In Zilsel's case, specializing in sociology, the method supposed to lead to the goal was sociological and historical comparison, not Neurath's "physicalism".

Zilsel was marginal in the Vienna environment. He remained so after his post-*Anschluss* emigration to the U.S., where he was associated to the International Institute of Social Research, the emigrated version of the Frankfurt Institut für Sozialforschung [Raven & Krohn 2000: xxi]. After his suicide in 1944, he was almost forgotten (the history of science constituting a partial exception) – in particular he disappeared from historical accounts of logical empiricism, at least until recently.[2]

During his stay in the U.S. he worked (until 1941 on a Rockefeller grant, then in the scarce time left over from earning a living) on a project on the social origins of Modern science. The articles communicating partial and preliminary results from this project have secured him some fame among historians of science.[3]

First of these was a paper on "The Sociological Roots of Science" [1942]. Its abstract runs as follows:

> In the period from 1300 to 1600 three strata of intellectual activity must be distinguished: university scholars, humanists, and artisans. Both university scholars and humanists were rationally trained. Their methods, however, were determined by their professional conditions and differed substantially from the methods of science. Both professors and humanistic literati distinguished liberal from mechanical arts and despised manual labor, experimentation, and dissection. Craftsmen were the

[1] As Carnap [1932] gave up genuine empiricism with his interpretation of the concept of "protocol sentences" (whose relation to some real world was considered outside the philosopher's field), Zilsel was the first to attack him [1932].

[2] Things may perhaps be changing. In K. Mulligan's three-page article [2001] on "Logical Positivism and Logical Empiricism" in *International Encyclopedia of the Social & Behavioral Sciences*, Zilsel and Neurath have four items each in the bibliography, Carnap three, and nobody else more than two; but the context is of course one where importance for genuine empirical research counts. In [Hardcastle & Richardson 2003], Zilsel gets a whole chapter [Raven 2003].

[3] Now published collectively in [Raven & Krohn 2000].

> pioneers of causal thinking in this period. Certain groups of superior manual laborers (artist-engineers, surgeons, the makers of nautical and musical instruments, surveyors, navigators, gunners) experimented, dissected, and used quantitative methods. The measuring instruments of the navigators, surveyors, and gunners were the forerunners of the later physical instruments. The craftsmen, however, lacked methodical intellectual training. Thus the two components of the scientific method were separated by a social barrier: logical training was reserved for upper-class scholars; experimentation, causal interest, and quantitative method were left to more or less plebeian artisans. Science was born when, with the progress of technology, the experimental method eventually overcame the social prejudice against manual labor and was adopted by rationally trained scholars. This was accomplished about 1600 (Gilbert, Galileo, Bacon). At the same time the scholastic method of disputation and the humanistic ideal of individual glory were superseded by the ideals of control of nature and advancement of learning through scientific co-operation. In a somewhat different way, sociologically, modern astronomy developed. The whole process was imbedded in the advance of early capitalistic society, which weakened collective-mindedness, magical thinking, and belief in authority and which furthered worldly, causal, rational, and quantitative thinking.

Summing up the summary, neither the university tradition nor Renaissance Humanism nor technicians created the scientific revolution on its own – what was decisive was the *interaction* between,the mutual fecundation of the three groups. True to his comparativist methodology, Zilsel ends by pointing to the problem that

> In China, slave labor was not predominant, and money economy had existed since about 500 B.C. Also there were in China, on the one hand, highly skilled artisans and, on the other, scholar-officials, approximately corresponding to the European humanists. Yet causal, experimental, and quantitative science not bound to authorities did not arise. Why this did not happen is as little explained as why capitalism did not develop in China.

In several of the articles collected in the volume *The Grand Titration*, Joseph Needham [1969] shows his inspiration from Zilsel – in particular “Time and Eastern Man” (pp. 218–298), originally a lecture from 1964. Even elsewhere – e.g. in the title *Clerks and Craftsmen in China and the West* [1970] – his debt shows up (evidently, for comparative purposes the scholastic university tradition has to be left out, having no separate analogue in China).

Zilsel’s ideas – together with Boris Hessen’s and Robert Merton’s work on seventeenth-century England – have inspired other workers to agreement or debate; I shall only mention [Hall 1959] and (some of) the articles collected in [Field & James 1993]. My intention here is to see how far the idea can be applied to a parallel field which neither Zilsel not the discussions after his time have taken up: the emergence of “Modern algebra” (that of the late sixteenth and earlier seventeenth century, to be distinguished from "modern algebra", that of the last century).

The three acting groups

Of Zilsel's three groups, Renaissance Humanists can be taken over directly into our story. At the global level, university scholars also recur; however, what is interesting for us is not the natural philosophers of Merton College and their kin but the readers of Euclid and of other ancient mathematicians; for this reason, it is worthwhile to include also the pre-university translators of the twelfth century. Artisans, finally, are not to be understood as gunners and master builders but as the masters of the Italian abbacus school. Since not all groups were active at the same time, a mainly chronological ordering of the argument will be fitting.

Latin twelfth- to thirteenth-century reception

Algebra first appeared in Latin in four twelfth-century works. The fragment in the "Toledan *Regule*" (the second part of the *Liber alchorismi de pratica arismetice*) [ed. Burnett et al 2007: 163–165] was without the slightest influence. So was almost certainly the presentation of the technique in a chapter in the *Liber mahamaleth* [ed. Vlasschaert 2010] from around 1160: the chapter in question is absent from all three manuscripts, we only know about it from blind cross-references in other parts of the work. Various problems solved by means of algebra have survived in the manuscripts; however, they are so different from anything else we find that we may safely conclude that even they had no impact.

Remain the two translations of al-Khwārizmī's algebra. One was made by Robert of Chester [ed. Hughes 1989] in c. 1145, the other by Gherardo da Cremona [ed. Hughes 1986] in c. 1170. They build on different Arabic manuscripts, the one used by Gherardo being closer to the lost original than that used by Robert (and apparently also closer to the original than the extant Arabic manuscripts, see [Høyrup 1998] and [Rashed 2007: 88–90]). Since Robert's use of *substantia* as the translation of Arabic *māl* found no echo, even *his* translation seems to have had scarce influence – Gherardo uses *census*, also found in the anonymous *Liber augmenti et diminutionis* [ed. Libri 1838: I, 304–376 + Hughes 2001] and thus perhaps the established translation in 1170, in any case the term chosen by Fibonacci (who copies from Gherardo [Miura 1981]) and later used in the Italian tradition (as *censo*). The same conclusion follows from the distribution of manuscripts: copies of Gherardo's translation come from many places, whereas all three extant manuscripts of Roberts translation were made in fifteenth-century southern Germany.

So, algebra was essentially received through Gherardo's translation, which is very faithful to al-Khwārizmī's original.[4] We find in it:

- the rules for solving the six simple and composite equation types ("cases") of the first and second degree. The rules are formulated for the normalized equations (except for the type "roots are made equal to number", in which case the normalized equation *is* the solution), but numerical examples show how to reduce non-normalized equations to normalized form. The algebraic unknown is a *census*[5] and its (square) *root*.
- Geometrical demonstrations for the correctness of the rules for the composite cases.
- An explanation of how to multiply additive or subtractive binomials (involving two integer or fractional numbers or an algebraic *res* / "thing" and a number).
- An explanation of the addition and subtraction of bi- and trinomials involving numbers, square roots of numbers, *census* and / or algebraic *roots*. For binomials a geometric argument is given (drawing along two mutually perpendicular axes), for trinomials al-Khwārizmī says that he tried to make a similar proof, but it was not clear – "but its necessity is obvious from the words".[6] Between the explanations and the proofs, it is taught how to multiply and divide square roots.
- Six problems illustrating the six rules. Here, the algebraic unknown is first labelled *res*, but its product with itself is then identified with the *census*, the *res* thereby becoming a *root*.
- A section on miscellaneous problems, also illustrating the use of the rules. This section contains more problems in Robert's version and in the Arabic manuscripts (not always the same number).

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[4] Apart from the occasional use of Hindu-Arabic numerals and the inclusion of a number of extra problems that had crept into the Arabic tradition after al-Khwārizmī's time, even Robert's translation is actually faithful to the original. Even if it had been more popular, it would not have given the Latin world access to the more recent transformations of Arabic algebra.

[5] Translating Arabic *māl*, originally (and still today in general language) a sum of money, but more or less reduced to a formal magnitude, characterized solely by being the product of its (square) root by itself.

[6] My translation, as all translations in the following where no translator is identified.

– A chapter on “merchants’ agreements”, actually about the rule of three (but this name is not stated, nor was it used by al-Khwārizmī).
– An appendix with problems “found in another book”, namely some of the miscellaneous problems known from the extant Arabic manuscripts but absent from the one used by Gherardo.

The chapters on geometrical calculation and on inheritance arithmetic are left out by Gherardo (if not by his source manuscript).

Even Gherardo’s translation only had no strong impact, and for good reasons. There were, roughly speaking, two motives for the translation of philosophical and scientific works from the Arabic. One was the desire to get hold of those purportedly central works that were known by title only from such encyclopedic works as Martianus Capella’s *Marriage of Mercury and Philology*. This, of course, could not concern al-Khwārizmī and algebra. The other I have termed “medico-astrological naturalism”,[7] and had astronomy subservient to astrology as an essential ingredient (together, of course, with medicine and astrology *stricto sensu*). For those in direct contact with the Arabic tradition, it would be known that al-Khwārizmī’s algebra was reckoned among the “middle books” (together with Euclid’s *Data* and various works on spherics), the books that were to be read between the *Elements* and the *Almagest* [Steinschneider 1865], and it would therefore be an obvious choice to translate it, just as al-Khwārizmī’s introduction to the Hindu-Arabic numerals was translated as an essential tool for astronomical table-making and calculation. But whereas Hindu-Arabic numerals really served in astronomy, algebra did not serve astronomy as it came to be practised in Latin Europe in any way – at least not before Regiomontanus, inspired by Jābir ibn Aflaḥ, applied it in the 1460s when proving certain trigonometrical theorems. In consequence, few university scholars had any reason to take up the topic.

The 1228-version of Fibonacci’s *Liber abbaci* [ed. Boncompagni 1857] contains a final section on algebra, and there is no reason to believe a similar section was not present in the first version from 1202, perhaps with fewer problems. As pointed out by Miura Nobuo, Fibonacci draws to some extent on Gherardo in the beginning of this presentation, but afterwards many problems are solved which are not borrowed from al-Khwārizmı. Some share the mathematical structure and the parameters (but not the structure of the formulation) with Abū Kamil, others only the mathematical structure, and some share their structure

[7] I introduced the term in [Høyrup 1987: 30] (reprint [Høyrup 1994: 140]. A more thorough discussion of its various aspects and its role in the “translation movement” can be found in [Høyrup 2004: 185–187].

with al-Karajī's *Fakhrī*. Comparison of Fibonacci's text with the original in cases where we know that he copies (namely from Gherardo's translations of al-Khwārizmī's algebra and of Abū Bakr's *Liber mensurationum*[8]) shows that he did not try to conceal his borrowings; we may therefore conclude that he did not use Abū Kāmil's and al-Karajī's books directly but drew on what circulated somewhere in the Arabic world in his own times. However, there are no traces of the algebraic symbolism that had been developed in the (presumably later) twelfth century in the Maghreb, nor of al-Karajī's elaboration of a theory of polynomials or his approaches to a purely algebraic proof technique.

Fibonacci still gives geometric proofs of the rules for solving the mixed second-degree equations. For the case "*census* and *things* are made equal to number" he even gives two, just like al-Khwārizmī, but not the same; the first corresponds to the underlying idea of *Elements* II.7 (al-Khwārizmī's second proof is a similarly "naive" counterpart of *Elements* II.6), while the second builds on the formulation of *Elements* II.6 (yet without mentioning this source, which Fibonacci is otherwise fond of parading).[9] Even later in the algebra chapter, geometric proofs abound, whereas there are none in al-Khwārizmī.[10] For Fibonacci, *proof* was *geometric proof*, in agreement with his orientation toward Greek theory.

Well before introducing algebra explicitly, Fibonacci makes use of a technique which *we* cannot avoid recognizing as rhetorical algebra of the first degree, but which Fibonacci conspicuously regards as something different. The unknown is designated *res*, but the method itself is spoken of *regula recta*, "direct rule".[11] The distinction is not Fibonacci's invention, the *Liber augmenti et diminutionis*[12] often offers an alternative solution by *regula* – which is exactly Fibonacci's *regula recta*. As far as I know, Fibonacci's *regula recta* never turns up in later Latin

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[8] See [Miura 1981] and [Høyrup 1996: 55].

[9] Both types were current in Arabic algebra of the epoch; the latter had already been introduced by Thābit ibn Qurrah [ed. Luckey 1941], the former is described (in an arithmetical version) by ibn al-Hāʾim [ed. Abdeljaouad 2004: 18*f*].

[10] In one passage [ed. Hughes 1986: 255; ed. Rashed 2007: 190–1293], al-Khwārizmī's reasoning presupposes a diagram, but he avoids speaking about it.

[11] [Ed. Boncompagni 1857: 191, and *passim*]. All occurrences are in chapter 12, containing mixed problems mostly of "recreational" type.

[12] [Ed. Libri 1838: I, 304–371], transformed into a critical edition by [Hughes 2001]. The work was translated in Iberian area (perhaps Toledo) during the twelfth century and is best known for introducing the method of a double false position in Latin mathematics.

mathematical writings as a name (nor does *regula*, with this meaning) – but without the name we shall encounter it below when discussing Jean de Murs. Benedetto da Firenze [ed. Arrighi 1974: 153, 168, 181], whom we shall also discuss below, refers to it as *modo retto/repto/recto* with unknown *quantità* – *his* source can hardly be Fibonacci or the *Liber augmenti* ... .

With one uncertain exception, we have no evidence that the *Liber abbaci* was read outside Italy before Jean de Murs did so in the mid-fourteenth century. Inside Italy, a number of copies and at least ono translation were produced during the next three centuries – some 15 still survive.

The possible exception is Jordanus de Nemore. Perhaps in the later 1220s,[13] he wrote the treatise *De numeris datis* [ed. Hughes 1981]. It emulates the format of Euclid's *Data* and applies it to the arithmetical domain. It is deductively organized and contains propositions of the form "if certain arithmetical combinations of certain numbers are given, then the numbers themselves are also given",[14] and formulates the proofs in an abstract letter symbolism. Jordanus does not mention algebra at all, but he gives numerical examples that often coincide with what can be found in corresponding problems in properly algebraic works, leaving no doubt that he had undertaken to reformulate algebra as a demonstrative arithmetical discipline, intentionally leaving so many traces that those who knew algebra would recognize the endeavour.

In many cases, the numerical examples coincide with those of al-Khwārizmī. In others, they point to either Abū Kāmil or Fibonacci [Høyrup 1988: 310 n.10] – and since the known Latin translation of Abū Kāmil's algebra was made in the fourteenth century [Sesiano 1993: 315–317], Jordanus is likely to have known the *Liber abbaci*.[15] A further suggestion (nothing more) in the same direction

[13] The treatise is written after his *De elementis arithmetice artis*, to which it refers, and the latter after the second version of the algorism treatises. Here, indeed, the letter symbolism is first developed which was then used to the full in the *De elementis*. One of the algorism treatises is copied (apparently by Grosseteste) in 1215/16 [Hunt 1955: 133*f*].

[14] For instance, I.17, "When a given number is divided into two parts, if the product of one by the other is divided by their difference, and the outcome is given, then each part will also be given". And IV.9, indicating the existence of a double solution to what we would express $x^2+b = ax$, "a square which with the addition of a given number makes a number that is produced by its root multiplied by a given number, can be obtained in two ways".

[15] On the other hand, we know that Jordanus knew aspects of Arabic mathematics for which we have no idea about his sources. Furthermore, the lost algebra chapter of the *Liber mahamaleth* may still have existed in the early thirteenth century.

comes from what Jordanus presents in II.27 as "the Arabic method" to solve the problem of the "purchase of a horse", which has some similarity (namely in the parameters) to what we find in the *Liber abbaci* [ed. Boncompagni 1857: 245–248].

De numeris datis is not a mere reformulation. The quest for deductivity as well as Jordanus's general inclinations cause the outcome of his undertaking to be at least as much of a piece of coherent theory as the Euclidean model. It also goes beyond what was done in Arabic algebra – book II, starting with the equivalent of the rule of three (thus the equivalent of the final chapter of Gherardo's translation of al-Khwārizmī's algebra) develops into a wide-ranging investigation of proportion theory. Book III contains further elaboration of the same topic.[16]

A small circle seems to have existed around Jordanus, comprising Campanus and Richard de Fournival while being at least known to Roger Bacon [Høyrup 1988: 343–351]. It is often claimed that *De numeris datis* became the standard algebra textbook in the scholastic university. Unfortunately, there is no documentary basis whatsoever for assuming that there *was* any algebra teaching there, and *a fortiori* not for assuming that Jordanus's treatise served. What we know from the fourteenth century is that Oresme cites the *De elementis* and the *De numeris datis* in three of his works[17]. Oresme being without competition the foremost Latin mathematician of his century, *his* use of another eminent mathematician proves little concerning his contemporaries.

In the fifteenth century, two famous Vienna astronomers demonstrate that they not only knew the work but also understood in what way it was related to the Arabic art and in which way it differed. One is Georg Peurbach, who in a poem [ed. Größing 1983: 210] refers to "the extraordinary ways of the Arabs, the force of the entirety of numbers so beautiful to know, what algebra computes, what Jordanus demonstrates". The other is Regiomontanus, in whose Padua lecture on the mathematical sciences from 1464 [ed. Schmeidler 1972: 46] we read

[16] A few of the propositions from book III coincide with what can be found in chapter 15 part 1 of the *Liber abbaci* (e.g., III.14 and III.15). However, the two contexts are so different – both of them systematic but organized according to different principles – that this coincidence is likely to be accidental.

[17] The former in *Algorismus proportionum* [ed. Curtze 1868: 14], in *De proportionibus proportionum* [ed. Grant 1966: 140, 148, 180] and *Tractatus de commensurabilitate vel incommensurabilitate motuum celi* [ed. Grant 1971: 294] (merely a complaint that Jordanus's subtle work is inapplicable to the presumably irrational ratios of celestial speeds); the latter in *De proportionibus proportionum* [ed. Grant 1966: 164, 266] (references to propositions about elementary proportion theory).

about the “three most beautiful books about given numbers” which Jordanus

> had published on the basis of his *Elements of arithmetic* in ten books. Until now, however, nobody has translated from the Greek into Latin the thirteen most subtle books of Diophantos, in which the flower of the whole of arithmetic is hidden, namely the art of the thing and the *census*, which today is called algebra by an Arabic name.

The reference to Diophantos anticipates Regiomontanus’s interaction with the Humanist current; for the moment we may just take it as another way to specify the relation between Jordanus’s treatise and the algebraic discipline.[18]

In the list of books left by the later less famous Vienna astronomer Andreas Stiborius in c. 1500 we find as neighbouring items Euclid’s *Data*, Jordanus’s *De numeris datis* and *Demonstrationes cosse* (an unidentifiable work on algebra in Italian-German tradition) [Clagett 1978: 347]. Either Stiborius or Georg Tannstetter (who made the list) thus understood *De numeris datis* as belonging midway between Euclid’s *Data* and algebra.

Jordanus was certainly an eminent representative of the universitarian mathematical environment, even if his work had little impact on the further development of university mathematics. Fibonacci is less easy to categorize. He wrote in Latin; much of the material he presents is similar to what we find later in the abbacus tradition; he often applies methods belonging to the “scientific” mathematical tradition, in particular geometric reasoning in Euclidean style, and refers to abbacus-type methods as “vernacular” (using terms like *vulgariter*);[19] his reception within the university tradition was very limited; and, as we shall see, so was his impact on the abbacus tradition. It is thus impossible to locate him within any of the three traditions; what we can say is that he is a witness of the existence of something close to the later abbacus tradition already around 1200, and that his book is a first attempt at synthesis between the practical and the scholarly tradition[20] – a heroic but premature attempt, and in consequence a heroic failure, we may say.

[18] We also know that Regiomontanus listed *De numeris datis* in the leaflet containing his publishing plans (which were never realized because of his sudden death) [ed. Schmeidler 1972: 533]. However, all this text tells us (albeit indirectly, by including it) is that Regiomontanus considered the treatise important.

[19] [Ed. Boncompagni 1857: 63, 11, 114, 115, 127, 170, 204, 364].

[20] See [Høyrup 2005; 2009a].

The fourteenth century – early abbacus algebra, and first interaction

Abbacus teachers and schools are first mentioned in the sources from 1265 onward.[21] These schools trained merchants' and artisans' sons (and often sons of the urban patriciate) for two years or less around ages 11–12. They thrived between northern Italy (not least Genua, Milan and Venice) and Umbria until the mid-sixteenth century.

The curriculum, as we know it from one explicit description and one contract between a master and an assistant[22] (and as confirmed by scattered remarks in various abbacus books) encompassed the following:

- First the practice of numbers: writing numbers with Hindu-Arabic numerals; the multiplication tables and their application; division, first by divisors known from the multiplication tables, then by multi-digit divisors; calculation with fractions.
- Then topics from commercial mathematics (in varying order in the two documents): the rule of three; monetary and metrological conversions; simple and composite interest, and reduction to interest per day; partnership; simple and composite discounting; alloying; the technique of a "single false position"; and area measurement.

Everything, from the multiplication tables onward, was accompanied by problems to be solved as homework. Manuscript books being expensive, the teaching was evidently oral. The "abbacus books" written by many teachers were thus not meant as textbooks for the school. Some were written explicitly as gifts to patrons or friends, some perhaps as teachers' handbooks (that is at best an educated guess), some claim to be suited for self-education. They often include topics that go beyond the curriculum, such as the double false position and algebra. These may have served in the training of assistant-apprentices, but this is another speculation with no support in the sources; in any case we know that proficiency in such difficult matters was important in the competition for employment (smaller towns often employed an abbacus teachers) or for paying pupils.

The earliest two abbacus books are from Umbria from the outgoing thirteenth century (both known, however, from fourteenth-century copies). One of them (the "Columbia algorism", ed. [Vogel 1977]), apparently written shortly after 1278 [Høyrup 2007a: 31], reveals some puzzling affinities with Iberian fourteenth-

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[21] Apart from her unsubstantiated belief in the inspiring role of Fibonacci, Elisabetta Ulivi's overview [2002: 124–126] of the beginnings of the institution can be recommended.

[22] The former (describing the Pisa curriculum) from the 1420s [ed. Arrighi 1967a], the latter from Florence from 1519 [ed. Goldthwaite 1972: 421–425].

century material. The other, the *Livero de l'abbecho*, is probably somewhat but not very much later [Høyrup 2007a: 34]. The latter claims in its introductory lines to be written "according to the opinion" of Fibonacci. As close analysis shows, the treatise falls into two parts [Høyrup 2005]. One, elementary and corresponding to the curriculum of the school, borrows nothing at all from Fibonacci; the other consists almost exclusively of sophisticated problems borrowed from Fibonacci – but demonstrably borrowed without understanding, and without the compiler having followed the calculations.[23] Obviously, Fibonacci cannot have inspired the actual teaching of the compiler: his role is that of prestigious decoration.

Neither the Columbia algorism nor later ordinary abbacus treatises owe more to Fibonacci (we shall return to three apparent exceptions from the mid-fifteenth century) – actually, they do not even refer to him. We must conclude that the tradition did not (as often claimed without the slightest support in the sources) derive from Fibonacci's *Liber abbaci* and *Pratica geometrie*. It had its roots in the larger Mediterranean tradition for commercial calculation – in Arabic *muᶜamalāt* mathematics, but probably in particular in Iberian practices. Fibonacci had been acquainted with the same practices a small century earlier, but by presenting what he had learned from them according to scholarly norms he had efficiently barred diffusion to the mathematically humble abbacus teachers.

Further details about the origin of the abbacus tradition do not concern us here. All we need to notice is that it existed as an independent tradition.

Algebra was no part of the early abbacus tradition – the compiler of the *Livero de l'abbecho* demonstrates by occasional misunderstandings of Fibonacci's words that he has never heard about it. The earliest abbacus algebra is likely to be the one contained in Jacopo da Firenze's *Tractatus algorismi*, written in Montpellier in 1307.[24]

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[23] In particular, Fibonacci's notation for ascending continued fractions, used profusely in the *Liber abbaci* (including the parts copied by the Umbrian compiler) are understood as ordinary fractions.

[24] See [Høyrup 2007a]. All three manuscripts are probably fifteenth-century copies: Vatican Vat. Lat. 4826 (**V**) can be dated by watermarks to c. 1450. Milan, Trivulziana 90 (**M**) in the same way to c. 1410. Florence, Riccardiana 2236 (**F**) is written on vellum and hence carries no watermarks. However, it is closely linked to **M** but slightly more corrupt; if not necessarily in date then at least in distance from the archetype it is therefore later than **M**.

F and **M** are in any case closely related, and apparently descendants of a revision leaving out part of the original treatise so as to fit it to the school curriculum. Only **V**

This algebra is very different, both from that of the *Liber abbaci* and from anything we know (in the original language or in translation) from the hands of al-Khwārizmī, Abū Kāmil and al-Karajī (although it has more in common with al-Karajī's elementary *Kafī* than with his advanced *Fakhrī* and *Badī*ᶜ and with the other two authors). Its descent from Arabic algebra is indubitable, and the use of the term *census* (actually Tuscan *censo*) for *māl* is shared by various Iberian twelfth-century translations.

Jacopo first presents rules for the six basic cases (those of the first and second degree), already dealt with by al-Khwārizmī. These are provided with examples. Then follow fourteen that can either be solved by simple root extraction or reduced to one of the initial six examples.They are not followed by examples.

The "root" has disappeared from the rules, being everywhere replaced by the "thing" (Tuscan *cosa*), and all rules are formulated so as to cover non-normalized equations.[25] More significant, all references to geometric proofs have disappeared. Further, the term *raoguaglamento*, probably translating Arabic *muqābalah*, is used about the confrontation of the two sides of an equation, in agreement with the literal meaning of this Arabic term and probably with its original technical use [Saliba 1972]; from al-Khwārizmī onward, however, with al-Karajī as the sole exception, the term was habitually used to designate the simplification of an equation by removal of additive terms.[26] *Ristorare*, corresponding to Arabic *jabara* (whence *al-jabr*, the term that was Latinized as *Algebra*) and in most Arabic sources referring to the elimination of a subtractive term through addition, is used by Jacopo for additive as well as subtractive simplification.

Finally, Jacopo's examples not only differ in actual content from those encountered in al-Khwārizmī (etc.) and the *Liber abbaci*, many of them also differ

contains the algebra section, which could therefore well be a late insertion. However, stylistic considerations strongly suggest that it was written by the same hand as the archetype for all three manuscripts. In any case, comparison with other abbacus algebras from the early phase shows that it reflects the character of the discipline at the moment of reception, and that it must belong to the early decades of the century. For convenience, I shall speak of its date as 1307 and of its compiler as Jacopo.

[25] That is, the first step is a division by the coefficient of the *censi*. In contrast to what is done by al-Khwārizmī (and, in different words, still in the *Liber mahamaleth*), no distinction is made between divisors greater than respectively smaller than 1.

[26] In al-Khwārizmī, however, the term seems rather to refer to the *production of a simplified equation* via such subtraction on both sides – which may eventually have led to the change of meaning.

in character. Those of al-Khwārizmī and Fibonacci (and of Abū Kāmil too) are either pure-number problems or, at most, deal with an unspecified "capital" or with an amount of money divided between a number of men. Half of Jacopo's ten examples pretend to deal with real commercial problems – and one with a square root of *real* money, not merely a formal *māl*. Commercial problems, we may observe, abound in ibn Badr's *Ikhtisār al-jabr wa'l-muqābalah* [ed. Sánches Pérez 1916], *possibly* of Iberian origin, and square roots of real money in the *Liber mahamaleth*.[27]

The further development of algebra built on this foundation – not all of it directly on Jacopo, but in any case on the same source tradition.[28] Within a couple of decades, however, new elements were added, presumably borrowed directly or indirectly from what had been developed in the Maghreb and/or al-Andalus (Muslim Spain) in the twelfth century [Høyrup 2008]: calculation with "formal fractions" (e.g., $\frac{100}{1\ cosa}+\frac{100}{1\ cosa\ +\ 5}$); (mostly unsystematic) use of abbreviations for *root*, *cosa* and *censo*; and the use of schemes for the calculation with binomials. More problematic, and probably no borrrowing but a local development, the fields becomes infested with false solutions to irreducible third- and fourth-degree cases,[29] surviving into the mid-sixteenth century (where Bento Fernandes copied them in his Portuguese *Tratado da arte de arismetica* [Silva 2008: 11]).

Some authors understood the false solutions *were* false. In 1344, master Dardi da Pisa wrote the earliest extant treatise dedicated exclusively to algebra. He solves no less than 194 cases correctly – a huge number he attains by including

[27] A more elaborate discussion of the distinctive characteristics of Jacopo's (and subsequent abbacus) algebra can be found in [Høyrup 2007a: 156].

[28] Since Jacopo's treatise contains no single Arabism, this tradition must have been located in a (non-Italian) Romance-speaking area – most plausibly in Catalonia or the Catalan-Provençal area, as argued in [Høyrup 2007a: 166–182]. But this is of no concern here.

[29] Solving for instance, the equation $\alpha K = \beta c+\gamma$ as if it had been $\alpha Ç = \beta c+\gamma$ (K stands for *cubo*, *Ç* for *censo*, *c* for *cosa*). Anybody with algebraic insight would discover that this can only be true if $K = C$, that is, if $c = 0$ (not admitted at the time) or $c = 1$ – which means that those who accepted the solutions either did not have this insight or expected their public not to posses it.

Control, we should notice, did not work well. The wrong solutions led to expressions involving radicals, and since abbacus algebra (in contrast to abbacus geometry) never stooped to approximation, it would have required hard work to find out they would not work.

complicated radicals (e.g., $\alpha c+\beta\sqrt{K} = \gamma Ç$), whose correct treatment shows that he understood the nature of the sequence of algebraic powers well. He also includes 4 rules for irreducible cases which only hold under special circumstances (as he says without specifying these); they are almost certainly not his own brew, but the one who derived them from obviously reducible cases by changes of variable[30] must have had a very good understanding of polynomial algebra.

Toward the end of the fourteenth century, a treatise from Florence containing a very long chapter on algebra.[31] Here, the nature of the sequence of algebraic powers as a geometric progression is set out explicitly, and it is shown how equations of the types $K+\beta Ç = m$, $K = \beta Ç+m$ and $\beta Ç = K+m$ can be reduced to the form $K = n+\alpha c$. The transformations are not explained in detail, but the transformed non-reduced coefficients show beyond doubt that the author makes the change of variable and the consecutive operations exactly as we would do it. We also find schemes for the multiplication of trinomials, modelled after the algorithm *a scacchiera* ("on chessboard") for multiplying multi-digit numbers.

Fifteenth-century copies of Antonio de Mazzinghi's late fourteenth-century writings show that *his* insights were even deeper. But they were also exceptional and had no impact of relevance for our theme, and we may leave them aside.

Fourteenth-century Humanism, as represented by such figures as Petrarca and Boccaccio, was purely literary. It did not make any attempt to approach mathematics, neither universitarian nor of abbacus type; nor did university mathematicians or abbacus masters take any interest in what they were doing.

One wellknown university mathematician, however, took up algebra, in part from Gherardo's translation of al-Khwārizmī, in part from the *Liber abbaci,* in part from familiarity with unidentified abbacus writings: Jean de Murs, in his mid-century *Quadripartitum numerorum* [ed. l'Huillier 1990], which in Regiomontanus's above-mentioned publishing prospectus stands alongside Jordanus's *De numeris datis*. Regiomontanus does not characterize it as an algebraic work, nor is it indeed one when taken as a whole. It consists of four books and a "half-book" (*semiliber*), book I of which is in a mixed Boethian-Euclidean tradition, whereas book II deals with calculation with the Hindu-Arabic numerals and with fractions. These two books are thus firmly rooted in the scholarly mathematical

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30 For instance, in a problem about a capital which grows from 100 £ to 150 £ in three years, taking the rate of interest as the unknown instead of the value of the capital after one year.

31 Florence, Biblioteca Nazionale, Fond. princ. II.V.152. [Franci & Pancanti 1988] is an edition of the algebra section .

tradition as it had been shaped from the twelfth century onward – the fraction-part of book II, however, rooted in twelfth-century works which we know from annotations to have been consulted by Jean[32] rather than in the university tradition, which (because Hindu-Arabic numerals served astronomy) was primarily interested in "physical" or "philosophical", that is, sexagesimal fractions.

Book III, the first to deal with algebra, is also in the schoolarly tradition. At first it takes up proportion theory (chapters 1–8); next follows an exposition of algebra, not copied from Gherardo but in its beginning closely depending on him – but omitting the geometric proofs. However, while writing this chapter Jean must have come across the *Liber abbaci*: the first three problems following after the general presentation are from al-Khwārizmī, but the rest are borrowed from Fibonacci, as shown by Ghislaine l'Huillier.

Between book III and book IV, Jean now inserts a *semiliber* or "half-book", stated to be an "explanation of what preceded and presentation of what comes".[33] Here, and also in book IV, the inspiration from the *Liber abbaci* is conspicuous – not only from its algebra section but also from chapter 12, containing mixed, to a large extent "recreational" problems. Even the *regula recta* turns up under the name *ars rei*, "the art of the thing" [ed. l'Huillier 1990: 418, 420*f*], mostly but not always in borrowed problems where Fibonacci already uses it. Jean also promises to propose many questions in book IV illustrating the method, but actually does not do so.[34]

But not everything in the *Quadripartitum* that is new to the school tradition comes from Fibonacci. Quite striking it the appearance and discussion of formal fractions [ed. l'Huillier 1990: 468*f*], for instance $\frac{\textit{res}}{10\ \textit{re diminuta}}$, that is, $\frac{\textit{thing}}{10\ \textit{diminished by a thing}}$.

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[32] Namely the *Liber algorismi* and a truncated copy of the *Liber mahamaleth*, both contained in the manuscript Paris, latin 15461 [l'Huillier 1990: XXX]. Both deal not only with ordinary fraction but also with "fractions of fractions", the *Liber mahamaleth* also with ascending continued fractions – both types customarily used in Arabic mathematics (and in the *Liber abbaci*). Jean takes up both types [ed. l'Huillier 1990: 204, 250].

[33] L'Huillier argues [1990: 13] that from the problem section of book III onward it is far from certain that things were written in the order they appear in the treatise. For our present purpose, this is immaterial.

[34] Non noticing that a particular method is spoken of, l'Huillier [1990: 13] believes that the questions referred to are those actually located at the end of book III, which is one of her arguments for doubting the order of the material. However, none of these problems make use of the art in question, for which reason this hypothesis must be rejected.

Jean even operates on them, adding (using our above abbreviations) $\frac{10-c}{c}$ and $\frac{c}{10-c}$ and finding the correct result $\frac{100-20c+2C}{10c-Ç}$ – precisely as done in contemporary advanced abbacus algebra. Besides that, we find systematic (but dubious) work on the products of algebraic powers and roots [ed. l'Huillier 1990: 463–469], going beyond what had been applied by al-Khwārizmī (but related to what Dardi must have known – and known better – a few years before). In addition, Jean uses the powers of 2 as an explanatory parallel to the algebraic powers – a device that was to be used by Rafaello Canacci [ed. Procissi 1954: 432] and by Luca Pacioli [1494: 143^r] in the late fifteenth century and which may have had fourteenth-century abbacus antecedents unknown to us.[35]

So, Jean adopts into a scholarly treatise material both from Fibonacci and from what was produced in his own times in the abbacus environment,[36] and attempts to subject it to the methodological norms of scholarly mathematics (not always with great success, he is no outstanding mathematician and tends to err when working on his own on difficult matters). But he does more. The methods by which "recreational" problems about pursuit are treated in book IV are applied afterwards to the astronomical problem of conjunctions (Jean was an eager practising astrologer no less than a mathematician, particularly interested in conjunctions – cf. [Poulle 1973: 131]). So, his aim is multi-faceted synthesis, not just incorporation.

According to Regiomontanus's prospectus, the work was "gushing with subtleties". Unfortunately (if we allow ourselves a disputable moral judgement of history) – better, unfortunately for Jean and his project, not many tended to see his work in that way, neither in his own nor in Regiomontanus's century. "Time was not yet ripe" – that is, those who had such interests were too rare to get into direct or indirect contact and to develop a common undertaking.

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[35] Another instance of use by Jean of abbacus material unknown to us is found in his *De arte mensurandi* [ed. Busard 1998: 187*f*], see [Høyrup 1999].

[36] Or, if we are to be (overly?) cautious, in some other area or surrounding, unknown to us, where similar undertakings were under way.

The fifteenth century – the beginnings of a *ménage à trois*

In the fifteenth century, some abbacus teachers took over norms both from the Humanist movement and from scholarly mathematics – some Humanists showed interest in mathematics (including abbacus mathematics) – and some mathematicians with university education and career took interest in "Humanist" (to be explained) as well as abbacus mathematics.

In the 1460s, three bulky "abbacus encyclopedias" were written in Florence, of which two – Benedetto da Firenze's *Praticha d'arismetricha* and the anonymous manuscript Florence, BN, Palatino 573[37] – show evidence of Humanist orientation.[38] Both authors, when writing on their own, are fully immersed in the abbacus tradition – specifically in a particular Florentine school tradition reaching back over Antonio de' Mazzinghi and the mid-fourteenth master Paolo dell'Abbacho to Biagio "il vecchio" († 1341). However, both also demonstrate Humanist interest in the foundations of their discipline. In the initial presentation of algebra, they choose not to base themselves on Fibonacci (whose problems they give later in separate chapters) or more recent authors from their school tradition (equally quoted at length with due reference) but on al-Khwārizmī (in Gherardo's version) – according to Benedetto because his proofs are *piu antiche* [ed. Salomone 1982: 20]. Their way to render Fibonacci's algebraic problems is also evidence of Humanistic deference to a venerated text – no changes are made (except translation, but both probably use a pre-existing Italian version), no new marginal commentaries are added, the margins only contain Fibonacci's own

[37] In a recently published paper, I maintain that the latter refers to the former as having been written "already some time ago" [Høyrup 2009b: 88]. This is an error, based in a misreading of a rather illegible manuscript copy, which I stupidly did not control in a partial transcription referred to in a note on the same page (!). The question of precedence cannot be decided, and since none of the two appears to borrow from the other (both, instead from shared identified sources), it is rather unimportant.

The third encyclopedia (Vatican, Ottobon. lat. 3307) shares many of the same sources but not the Humanist orientation.

All three are known from the respective authors' autographs, Benedetto's work also from several copies; his autograph is Siena, L.IV.21.

[38] It is sometimes difficult to distinguish internalized Humanist orientations from a strategy to impress aristocratic (actual or hoped-for) protectors. Since the two encyclopedists seem to write for "friends" on a rather equal footing (Benedetto even pointing out the mathematical failures of the recipient [ed. Arrighi 2004 / 1965: 138]), the orientation can be considered genuine.

diagrams. Even their respectful copying from predecessors in their school tradition points in the same direction.

But both also have ambitions to wrap their mathematics in scholarly garments. Book II (of 16) of Benedetto's treatise, dealing with "the nature and properties of numbers" is a presentation of speculative arithmetic in the Boethian tradition. It also offers an exposition and explanation of the complicated way ratios are named in this tradition (*doppia*, *sesquialtera*, etc.) [ed. Arrighi 1967b: 324*f*]. The first part of book V (on "the nature of numbers and proportional quantities") builds on the Campanus version of *Elements* V–IX and on Campanus's *De proportione et proportionalitate* about the composition of ratios (part 2 treats of metrological conversions). The first part of Book XI presents *Elements* II. The anonymous is less ambitious, but his chapter II.8 still deals with "the way to express as part, and, first, the definition" [Arrighi 2004/1967: 176], initially quoting Boethius's, Euclid's and Jordanus's definition of a ratio (*proportione*) as a relation between two numbers or quantities, going on afterwards with the Boethian names. This is not unproblematic; according to the definition a ratio is *not* a (possibly broken) number, as is the "part" the author wishes to express. He sees the difficulty but chooses to regard it as a mere question of language: "we in the schools do not use such terms [*vocaboli*] but say instead [...] that 8 is $^2/_3$ of 12 and 12 is $^3/_2$ of 8". He also points to the necessity that the two magnitudes in a ratio be of the same kind, but overlooks that this should create difficulties when, later, the concept is used to explain the rule of three.

This illustrates well the limited ambition (and actual reach) of this integration of abbacus and scholarly mathematics (Benedetto's as well as that of the anonymous): when it seems fitting, abbacus mathematics is put into the framework of scholarly mathematics, but the authors reinterpret concepts as needed, neglecting the contradictions that arise.

Luca Pacioli had similar aims, and in his case we also see them reflected in his biography: indeed, he rose socially from being an abbacus teacher to having the rank of a court mathematician (until Ludovico Sforza was driven out from Milan by the French) and to being a lecturer on and translator and editor of Euclid.

His *De divina proportione* – written while he was in Milan but printed in [1509] – is obviously inspired by Humanism (and by the wish to flatter the princely protector) in its long introduction (Pacioli is always longwinded) and elsewhere, and attempts to make mathematics a legitimate courtly-Humanist subject. But the translation of Piero della Francesca's *Libellus de quinque corporibus regularibus* which he includes in the printed edition makes use of algebra in abbacus tradition – and since Pacioli undertakes experiments with the symbolism, replacing Piero's conventional symbols with ◈ for *cosa* and ▫ for *censo*, algebra is not there simply because it was in Piero's original. If not a typographical play, these elegant shapes may perhaps represent an attempt to adjust to the artistic taste of the time (and hence to courtly-Humanist culture in a broad sense) and thus to make algebra more socially acceptable than traditional abbacus algebra would have been.

The *Summa de arithmetica, geometria, proportioni et proportionalita* from [1494] is different in orientation. The contents is primarily an encyclopedic presentation of abbacus mathematics. However, the authorities from whom Pacioli pretends initially to have borrowed most of the material are Euclid, Boethius, Fibonacci, Jordanus, Blasius of Parma, Sacrobosco and Prosdocimo de' Beldomandi[39] – all Latin writers (his Euclid is the Campanus edition), and all except Fibonacci bright stars on the heaven of university mathematics (but, excepting instead Euclid and Boethius, not exactly luminaries on that of contemporary Humanists). The work is thus (as also confirmed by the contents) in its general orientation a parallel to the *Liber abbaci*, submitting abbacus material to the norms of scholarly mathematics. The algebra to which Pacioli had access and which he presents is certainly much more sophisticated than what we find in Fibonacci – among unidentifiable others, Antonio de' Mazzinghi plays a role [Høyrup 2009b: 99]. But while Antonio may have understood at least in practice (we do not have his works in original, only through the three Florence encyclopedias) that purely algebraic demonstration was feasible, Pacioli stoops (like Fibonacci) to the idea that proof has to be geometric proof – apparently a regression if we look at matters in the perspective of the development of algebra as an autonomous branch of scientific mathematics, but perhaps less so if we think of Cardano's proof of the solutions for the cubic equations (see note 47).[40]

Among university mathematicians taking up algebra, the central figure is Regiomontanus. At least after coming in close contact with Bessarion in 1460–61, he clearly works intensely to connect mathematics with Humanist ideals. In his

[39] Borrowed he certainly has, but beyond Euclid and Fibonacci his sources are earlier abbacus masters.

[40] Nicolas Chuquet appears to have made a social move opposite to that of Pacioli, acquiring first a partial medical education. Exploring his work would lead far – too far for the present purpose, given that it resulted a dead end, Indeed, his only influence was through Etienne de la Roche, who borrowed freely from Chuquet's *Triparty* for his *Larismetique* from 1520, but excluded everything too radically new [Moss 1988: 120*f*].

Padua lecture from 1464, as we remember, he observed that until then "nobody [had] translated from the Greek into Latin the thirteen most subtle books of Diophantos, in which the flower of the whole of arithmetic is hidden, namely the art of the thing and the *census*, which today is called algebra by an Arabic name". He thus wants to understand algebra as a legitimately ancient and Greek art, or to make the audience see it thus.[41]

We also know the kind of algebra he practised when calculating in private, namely from his notes to the correspondence with Bianchini [ed. Curtze 1902: 192–336]. This is in the style of Florentine abbacus algebra of his own times – an illustrative example is discussed in [Høyrup 2010: 37*f*]. He also uses algebra occasionally in the *De triangulis* [ed. Hughes 1967] (II.12, II.23). Even this algebra is probably in abbacus style – but it is so simple that it might as well build on al-Khwārizmī as translated by Robert or Gherardo.[42] The Humanist connection certainly had no impact on his algebraic *practice*, neither inspiring transformation not preventing its use – nor could the problems to which Regiomontanus applied the technique ask for more than its traditional shape had to offer.

Another university-trained mathematician to be mentioned is a certain *magister* Gottfried Wolack, who held a university lecture in Erfurt in 1467 and again in 1468 [ed. Wappler 1900]. This lecture may have been the first *public* exposition of abbacus mathematics in German area, and may have played a role in legitimizing the field before the *Rechenmeister* tradition emerged – it still exists in several manuscripts and thus seems to have circulated well, and the idea of designating the *actors* of problems by the letters *A*, *B* and *C* (instead of "the first", "the second" and "the third" of three men) may have come from there.[43] However, "cossic" algebra (that is, Germanized algebra in abbacus style) already circulated in German manuscripts around a decade before, whereas Wolack has no algebra [Høyrup 2007b: 136–139].

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[41] In order to know that the work should have 13 books, Regiomontanus must have read at least (in) Diophantos's preface to book I. Since he believes all 13 books were actually present in the manuscript, he can not have inspected the whole of it. If after the preface he has only looked at book I, he will have had no occasion to discover that Diophantos mainly investigates indeterminate problems, and thus presupposes but hardly presents techniques similar to those of Arabic algebra.

[42] Albrecht Heeffer has inspected Regiomontanus's manuscript, which contains calculations in indubitable abbacus style (personal communication).

[43] At least I have noticed no earlier occurrence of the practice after years of looking and asking; the next time it turns up in an extant document seems to be in Christoph Rudolff's *Coss* [1525: N v^{r-v}].

Some Humanists, most notably Bessarion, already thought around 1460 that mathematics (but in particular mathematical astronomy) was an important part of the ancient legacy. The earliest Humanist of reputation to *take up* mathematics was probably Leone Battista Alberti. When looking at his treatises on perspective, in particular at his *Elementi di pittura* [ed. Grayson 1973: 109–129] we may find a merger of a broadly Humanist (but more precisely, artistic) and a mathematical outlook. Since this has nothing to do with algebra, I shall not pursue the question. His *Ludi rerum mathematicarum* [ed. Grayson 1973: 131–173] turns out not to offer more. Most of the work concerns elementary sighting geometry and area measurement. There is no trace of it being taken from contemporary abbacus geometry, even though that would have been possible. The authorities that are cited [ed. Grayson 1973: 153] are Columella and Savazorda [*sic*] among the ancients[44] and "Leonardo pisano among the moderns". Through its conception of mathematics as noble leisure, the work may have served to provide mathematics with Humanist legitimacy; but it did nothing for mathematics beyond that.

The first Humanist editions of ancient mathematical texts are Giorgio Valla's *De expetendis et fugiendis rebus* from 1501, where Euclidean material is scattered among other matters, and Bartolomeo Zamberti's translation of the *Elements* from an inferior Greek manuscript, printed in 1505. The former is a *florilège* and the second a mere text edition (made moreover, as pointed out by Maurolico, by a translator who knew Greek but was so far from being up to the topic that he did not discover the blunders of his inferior manuscript[45]).

Around 1500, at the very beginning of French Humanism, we also have Lefèvre d'Étaples mathematical editions. Their character is well illustrated by the purely medieval-quadrivial contents of the volume he brought out in [1496]:

– Jordanus's *Arithmetica decem libris demonstrata*;
– Lefévre d'Étaples' own *Elementa musicalia* in Boethian tradition;
– his *Epitome in duos libros arithmeticos divi Severini Boecii*;
– his description of *rithmomachia*, a board game invented around 1000 and serving to train the concepts of Boethian arithmetic.

[44] Since he can only have known Savasorda through Plato of Tivoli's translation, his sensibility to the "barbaric" Latin of the twelfth century cannot have been as acute as Humanists would like it to be.

[45] For instance in two letters *Illustrissimo Domino D. Ioanni Vegae* and *Illustrissimo Ac Reverendissimo Domino D. Marco Antonio Amulio* [Maurolico 1556; 1558]. See also [Rose 1975: 165].

No wonder, perhaps, that Humanism had had nothing to offer to mathematics in the previous century – a fortiori to algebra.

1500–1575: a changing scenery

After Pacioli's time, the abbacus environment *per se* was no longer theoretically productive in algebra – Cardano and Stifel, certainly working on algebra in the abbacus tradition, were scholars; Tartaglia, like Pacioli, worked hard and successfully to become one; Bombelli was an engineer-architect. Printed books linked directly to abbacus-like teaching (like [Borghi 1484], serving as "introduction for any youth dedicated to trade") tend to include no algebra (thus agreeing with the school curriculum). At most they would repeat what had been made before 1500 – like Ghaligai's *Summa de arithmetica*, reprinted in [Ghaligai 1552], the last chapter of which relates what Ghaligai had been taught about algebra by his master Giovanni del Sodo in the late fifteenth century.

The first to find the solution to certain irreducible third-degree equations – Scipione del Ferro, around 1505 – was a university professor, but his way to communicate it confidently to friends and students who could then use it in competitions shows vicinity to the abbacus norm system. However, since we have no knowledge about the deliberations that led him to the goal,[46] he is uninteresting for our purpose.

Let us therefore first look at a medical and intellectually omnivorous scholar who had turned his interest to abbacus-type mathematics, namely Cardano. Most of his mathematical writings have problems or methods from abbacus mathematics as their starting point. But they are written in Latin, and their shared aim is to produce scholarly mathematics, mostly in agreement with (some sort of) Euclidean norms. Further, he was versed in Humanist culture – this is already obvious from the language and the rhetorical content of his *Praise of Geometry*, read at the Academia Platina in Milan in 1535 [Cardano 1663: 440–445] (not to speak of his non-mathematical writings). That he implored Tartaglia to give him the solution to the cubic equations and then found the proofs (publishing them

[46] Masotti [1971: 596], following Giovanni Vacca, shows how mere experimentation with cubic binomials might suffice. Since such equations had been in focus since the earlier fourteenth century, and since it had been known by the more insightful abbacus algebraists for almost as long that the solutions that circulated were false, an attentive del Ferro may well have taken note if such play suddenly gave an interesting result.

with due reference once he discovered that Tartaglia had no priority)[47] does not distinguish him from what had been done in abbacus algebra at its best since centuries, nor that he went on and showed in the *Ars magna* [1545] how other mixed cubic equations could be reduced to these types (the necessary techniques were already used in the algebra contained in the manuscript spoken of in note 31). But Cardano went on from here to questions that had not been raised by any abbacus writer as far as we know, investigating the relation between coefficients and roots, and using the theory of irrationals of *Elements* X in order to find conditions which solutions would have to fulfil. He was not the first to work with negative numbers – Pacioli had done so, as well as the Florentine manuscript just mentioned. But Cardano did so more effortless that any precursor, and in the end of the book he even introduces their roots and operates with them – possibly because he had run into them when working on cubic problems, but actually on the basis of the second-degree problem $r+t = 10$, $rt = 40$, which everybody before him would just have dismissed as "impossible". A similar experimental spirit had been present in the university environment in Oresme's time, but certainly not after 1400. Nor was it common in fifteenth-century Humanism – Lorenzo Valla is the only exception that comes to (my) mind. But it was not foreign to the spirit that developed in the Humanism of the mid-sixteenth century – we may think of two works written at about the same time in Humanist style and famous in the history of science, Vesalius's *De fabrica humanis corporis* and Agricola's *De re metallica*. Both are respectful toward Antiquity – Agricola even shapes his title after Columella's *De re rustica* – but both are also delighted to follow tracks never explored by the ancients.

In 1545 and [1570],[48] it was possible for Cardano to pursue revolutionary novelties in algebra. Other famous writers in the field were less revolutionary. Stifel's *Arithmetica integra* from [1544] sets forth "all that was then known about arithmetic and algebra" [Vogel 1976: 59] and generalizes in a way his predecessors had not done (both in his development of polynomial algebra and in

[47] This is Cardano's version of the story; given his work on the problems to find two numbers from their product and either their sum or their difference in [1539] it sounds plausible – once you see the solution formulae for the equations $K+\alpha c = n$ and $K = \alpha c+n$, it is almost immediately clear that they have the corresponding structure, and from there the road to the geometric proof is also easy – in particular because of the importance such proofs had reacquired in algebra since Pacioli.

[48] Little has been done on the difficult *De regula aliza*, but Sara Confalonieri's ongoing project promises interesting insights.

his use of symbolism). He presents everything (or almost) developed or used by *some* Italian abbacus algebraist and deploys it systematically in a way *none* of them (including Pacioli) had done. We may claim that he brought the project of abbacus algebra to completion, as also recognized by those who borrowed from him – Tartaglia in Italy, Peletier in France.[49]

In 1535, Cardano had referred to Grynaeus's edition of the Greek Euclid with Proclos's commentary, published two years before. The *editio princeps* of Pappos's *Collection* appeared in Basel in 1538 (Commandino's Latin translation in 1588, after having circulated in manuscript), that of Archimedes in Basel in 1543; Memmo's Latin edition of books I–IV of Apollonios's *Conics* appeared in 1537 (Commandino's in 1566); Xylander's Latin translation of Diophantos was published in 1575 (the Greek *edition princeps* only in 1621). Only from the 1530s or 1540s is it thus possible to distinguish a genuine Humanist interest in mathematics. Maurolico's and Commandino's work in mathematics also began around this time.

However, being a mathematically interested Humanist was not sufficient to be able to contribute actively to the development of algebra. A good example is Peletier's *L'algebre* from [1554], which is decent but brings nothing new with respect to Stifel (in spite of Peletier's engagement in *linguistic* symbolization [1555]). Even being actively interested in Greek mathematics was not enough – here we may think of Buteo's *Logistica* from [1559]. A perfect precursor of Molière's *précieuses* (who also existed outside comedy), he finds the term *Arithmetica* too vulgar, and introduces *Logistica*. He writes ρ for the first power of the unknown (so had already Benedetto done, as well as the fourteenth-century manuscript referred to in note 31), ♦ for the second power and ❒ for the third, and P respectively M where Stifel, following [Widmann 1489], had used + and –; even these signs he may have considered vulgar since mercantile. His geometric proofs for the solution of the mixed cases refer explicitly to *Elements* II, and he adds and subtracts polynomials in schemes (as done in Italy since the fourteenth century). But he has no further theory, only problems, and none of his problems go beyond what could be found among fourteenth century abbacists. In all probability he had no intention to go beyond his predecessors, his aim may well have been to submit the elementary textbook genre to linguistic

[49] Ramus is of course an exception; in [1569: 66] he goes as far as to deny Stifel's very existence, which for somebody with Ramus's psychological constitution amounts to a confession that his algebra from [1560] depends (in all its poverty, and maybe indirectly) on the *Arithmetica integra*.

and notational purification (we may perhaps think of the father of Humanists, Petrarca, who would rather be ill than cured by Arabic-inspired university medicine).

Even Maurolico, a far better mathematician than Buteo and not burdened by linguistic prudishness, did little more in his short manuscript *Demonstratio algebrae* [ed. Fenaroli & Garibaldi, n.d.], and probably intended to do no more. The treatise is an orderly presentation of the sequence of algebraic powers as a geometric progression, with rules for multiplication and division. As had been formulated in many more words by Pacioli (and by other abbacists before him), Maurolico states that the traditional rules for the mixed second-degree cases can be used for all three-term equations where the middle power is "equidistant" (Pacioli's word, Maurolico makes no attempt to show off by speaking of geometric means) from the other two; and his geometric proofs refer to *Elements* II.

From the mid-sixteenth century onward, Boethius's *Arithmetica* and *De musica* gradually lost ground in university curricula, being replaced not by anything Humanist but rather by works linked to mathematical practice [Moyer 2001: 127–132]; but Humanism, with its emphasis on civic utility and civic leisure, may have contributed to preparing the ground for this (slow) change.

The take-off of Modern algebra

One effect of the full reception and initial creative work on the full Greek mathematical heritage was that *problems* moved into the focus of scholarly mathematics,[50] in contrast to the emphasis on *theory* of the high and late medieval Euclidean tradition.[51] This change of focus was due in part to what was found in the ancient texts themselves (not least Pappos), in part also to the kind of activity that came out of attempts to work creatively within the new theoretical framework provided by these texts – and of course also by the type of social interaction in which players like Viète, Fermat and Mersenne participated, competitive and communicative at the same time.

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[50] See [Bos 1996: 186–188] and, in general, [Bos 2001].

[51] Beyond the theologically tainted preference for the "speculative", the importance of which should perhaps not be overstated, the teaching style of universities probably played a major role in the creation of this emphasis. Lectures would allow the exposition of theory, and disputation invited metatheoretical reflections on the status and ontology of the discipline. Even written *quaestiones*, emulating the style of the disputation, invited philosophy of mathematics and not eristic work on problems.

But the change was not solely toward the solution of problems taken in isolation; it also implied interest in the general conditions for solvability and the general character of solutions – that is, in a new kind of theory. Inspiration for this theory and some answers (the classification of plane, solid and linear problems) could come from Pappos; but that did not suffice. Diophantos provided challenges rather than answers. Algebra, on the other hand, had always been primarily a technique for solving problems, and it already had some successes to exhibit within the kind of mathematics that now had the foreground. It will therefore have seemed obvious to re-investigate it in order to draw from it *not* just isolated problem solutions but also higher-level information.[52] The algebra

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[52] Descartes explains in the introduction to his *Discours de la Methode* [ed. Adam / Tannery XXX: 17f] (to which *La geometrie* is one of three appendices meant to test the new method) that he had hoped to get assistance for his project from logic, and, among the branches of mathematics, from "the analysis of the geometers and from algebra". But he immediately discards logic as an art which only serves to explain to others what one already knows, or even to speak of what one does not know. Analysis is too intimately bound up with the consideration of geometrical figures; algebra, finally has been so much "subjected to certain rules and certain signs that one has made out of it a confused and obscure art that puts the mind in difficulty instead of a science that cultivates it". Algebra thus seems to offer some hope, if only it could be liberated from these rules and signs – which is inded one of the things done in *La geometrie*.

In contrast, we may think of the more modest ambitions expressed by Pedro Nuñez in his *Libro de algebra en arithmetica y geometria*, published in [1567] but possibly written well before that year. Nuñez's aim is to show the wonders algebra can perform. However, in the geometry section we find statements like these (in total, he offers 77)):

– If the side of a square is known, the area will also be known;
– if the sum of the diameter and the side is known, each of them will also be known;
– if the side and the diameter and the area joined together, each of them will be known;
– if the product of the diameter and the area of the square is known, each of them will be known;
– if one of the sides [of a rectangle] and the diameter are known, the area will be known;
– if the area is known, and the two sides containing a right angle joined in one sum is known, each of the sides will be known;
– if the ratio of the two sides is known, and the magnitude of the diameter is known, or the ratio of the diameter to one of the sides as well as the other side are known, each of the others will be known;
– if the sides of a right triangle joined in one sum is known, and they are in [continued] proportion [...], each of the three will be known;
– if the two sides of a triangle are known, and the ratio between the parts of the base where the perpendicular falls is known, the base and the perpendicular and the area will be known;

that was taken over was not directly that of the abbacus masters but abbacus algebra as ordered and developed by Stifel and Cardano (and by Tartaglia, Nuñ2ez, Stevin and Clavius), and as it was also known through French writers like Peletier and Gosselin) – a creative synthesis drawing both on the abbacus tradition and on the meta-theoretical norms of high and late medieval university mathematics.

Evidently, Viète's reference to "a new art, or rather so old and so defiled and polluted by barbarians that I have found it necessary to bring it into, and invent, a completely new form" (Viète 1591: 2^v) is in itself a Humanist confession. But as demonstrated by Regiomontanus, such confessions might be mere lip service.[53] The mere wish to distinguish himself from the Arabs was certainly not what inspired his reformulation of the whole discipline, at most what induced him to use the terms *logistica*, *analysis*, *zetetics* and *poristics* – and it did not keep him from using also the term *algebra* albeit *nova*, instead of leaving it (like Jordanus) to readers to discover. Such rhetoric characterizes him as a scholar of Humanist constitution. But what caused his *mathematics* (and that of Descartes and Fermat, and others who did not contribute to the reshaping of algebra) to be Humanist, or rather post-Humanist, was their participation in an endeavour made possible (and next to compulsory for active theoretical mathematicians) by that relatively full access to the best ancient mathematicians that had been provided by sixteenth-century Humanism.

I shall not undertake a detailed analysis of the aims and the novelties of Viète's and Descartes' algebra – I would not be able to add anything of importance to Richard Witmer's "Translator's Introduction" [1983] nor to [Bos

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– if the three sides of the triangle are known, and a circled is described which touches its three sides, the semi-diameter of the circle will be known, and the parts of the sides divided by the touching points, and the distances from the centre to the angles of the triangle will also be known.

Only those about non-right triangles go beyond what we could find, for instance, in Abu Bakr's *Liber mensurationum* [ed. Busard 1968]. But we are still a far cry from some of the problems treated by Viète – for instance [trans. Witmer 1983: 403]

> If there are two individual isosceles triangles and the legs of one are equal to those of the other and the base angle of the second is equal to three times the base angle of the first, the cube of the base of the first minus three times the product of the base of the first and the square of the common leg is equal to the product of the base of the second and the square of the same leg.

[53] Actually, Clavius [1609: 4] quotes Regiomontanus' ascription to Diophantos as more verisimilar than the belief that the art be Arabic. But what is found in his book is quite in Stifel's style.

2001] (to name but these two). As an argument that the reformulation of the discipline was really needed for the post-Humanist mathematical project, and instead of drowning myself in a study of Fermat, I shall point to an episode that took place a small decade after the publication of Descartes' *Geometrie*. In 1645–46, the adolescent Huygens studied mathematics under the guidance of Franz van Schooten. Vol. 11 of his *Œuvres* contains a number of problems he investigated in this period by means of Cartesian algebra, many of which deal with matters inspired by Archimedes and Apollonios [Huygens 1908: 27–60]. Another sequence of problems (pp. 217–275), to be dated c. 1650, is partly derived from Pappos. It is difficult to imagine that they could have been efficiently dealt with by algebraic notations in Cardano's or Stifel's style.

Coda

As Diederick Raven and Wolfgang Krohn [2000: xxx–xxxiv] found out from inspection of Zilsel's unpublished papers, the thesis which inspired the present investigation was part of a larger project on "The social roots of modern science". In Zilsel's outline, mathematics only enters in section IV, "The rise of the quantitative spirit", subsection 2, "mathematics and its relation to commerce, military engineering, technology, and painting 1300–1600". Algebra is invisible. Zilsel's general thesis thus seems to receive unexpected and uninvited support from the creation of Modern algebra (still of course to be distinguished from "modern algebra", cf. above, p. 3.)

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